CAD&CG course

Masters in engineering

- **The course is given in 2024-2025 if there are enough students (5)**
- In that case, it will **not** be given in 2025-2026. Next session is 2026-2027

Please contact me if you are interested

E. Bechet

Eric Béchet (it's me !)

- Engineering studies in Nancy (Fr.)
- Ph.D in Montréal (Can.)
- Academic career in Nantes as Post-doc then ass. Prof. in Metz (Fr.)
  then Liège...

Alex Bolyn

- Assistant (Ph.D. student) at our university

Website **`http://www.cgeo.ulg.ac.be/CADCG`**

Contacts : **`eric.bechet@uliege.be`**
`a.bolyn@uliege.be`

Procedures

- Ex-Cathedra lectures (~2h)
- Practical sessions
  - Programming (~8 2h-sessions)
- Practical assessment : practical works (pw) are rated.
- Project (pr)
- The practical sessions, and the project are all mandatory. The final mark m=1/3pw + 2/3pr. , over 10 to pass & get the credits for the course.

Procedures

- Lectures are given on Wednesday, 1:45 PM – Lecture room is B52 +2/441 (or B52 +2/438 if few students)
- Begins today
- Practical work begins Sept. 25th after the course (check website for more dates & info)
- Any question : I am available mostly Fridays, appointment via email or phone : 04-366-9165 Office : Bldg B52 +2/438

Course Outline

- Introduction
  - Brief History of CAD Systems
- Generalities about Parametric Forms
  - Curves
  - Surfaces
- Some Practical Ways to represent Curves and Surfaces in CAD Systems
  - Cubic Splines
  - Bézier Curves
  - B-Splines

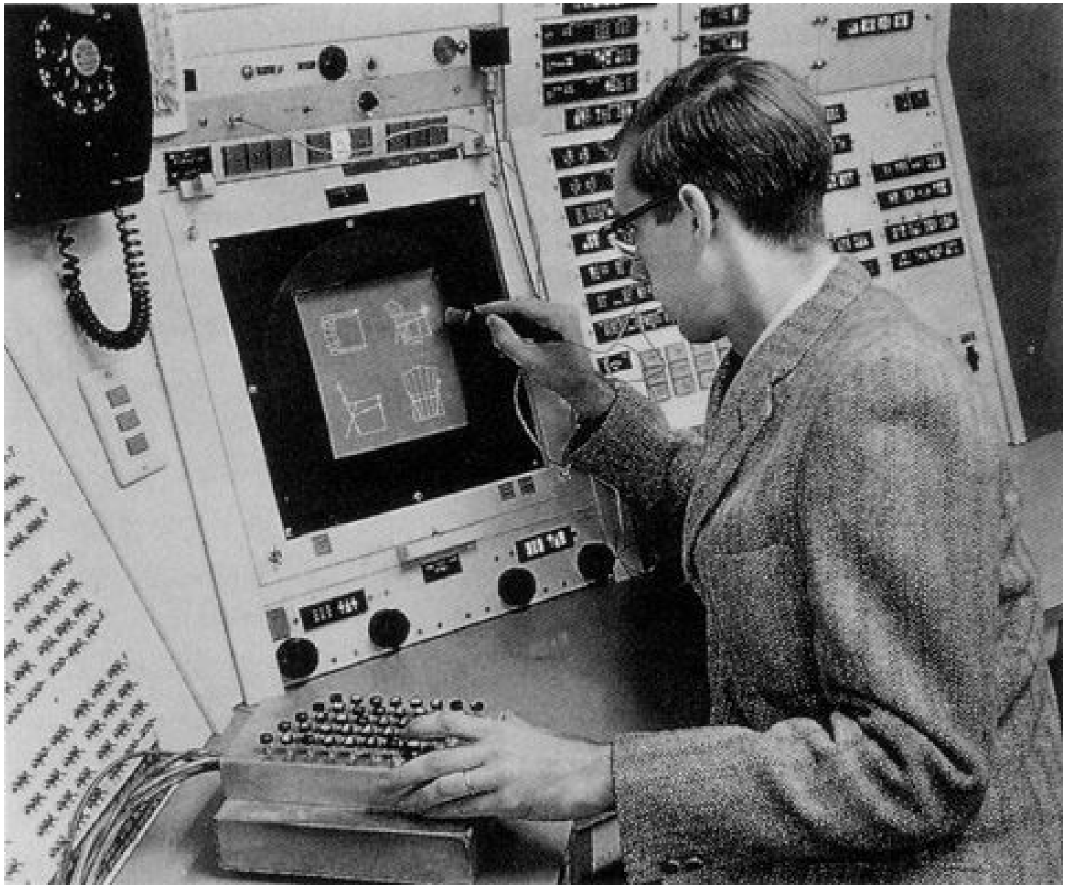
Course Outline

- Solid Modeling in CAD systems
  - B-REP Models
- Computational Geometry
  - Convex Hull Problems
  - Geometric Search
  - Delaunay Triangulation
  - Mesh Generation

CAD/CAM – Computer Aided Design / Computer Aided Manufacturing

- Development since the 1950s-1960s
  - Iso Schoenberg (at U-Wisconsin) – splines (40's)
  - First « interactive » CAD graphic engine at the MIT (I. Sutherland, Sketchpad) – 60's
  - James Ferguson (at Boeing) – splines (60's)
  - Paul De Casteljau (at Citroën) – Bézier curves (50's) and Pierre Bézier (at Renault) – " "
  - Carl De Boor (at GM / U-Wisconsin) – B-Splines (70's)
    - Catia, Unigraphics (now Siemens NX) date back to the mid-70's, emergence of 3d modeling kernels on mainframes.
- Huge expansion in the 80's with cheap personal computers becoming ubiquitous.

- Sketchpad (Ivan Sutherland) – 1963



Introduction

- Computational geometry
  - As old as ways to represent geometrical entities on a computer
  - Focus in this course
    - Use CAD as an initial geometrical database
    - Define operations in this setting and show how to perform them efficiently – or not !
    - Focus is made on applications in scientific computing, e.g. mesh generation, ...

Introduction

- Old mathematical and theoretical bases
- Practical framework has been developed because of industrial needs.
- In return, it has contributed at the theoretical level.

Parametric representation

Parametric representation

Use of scalar parameters to “walk” on the curve/surface

- The number of those scalars determine the dimension of the entity and the mapping from one space (the parametric space to the other (usually cartesian) space.
  - 0 – Point , 1 (u) – Curve , 2 (u,v) – Surface

    3 (u,v,w) – Volume (transformation used for finite elements)
- We associate to those scalars some function for each of the space coordinates (x,y in the plane; x,y,z in 3D)

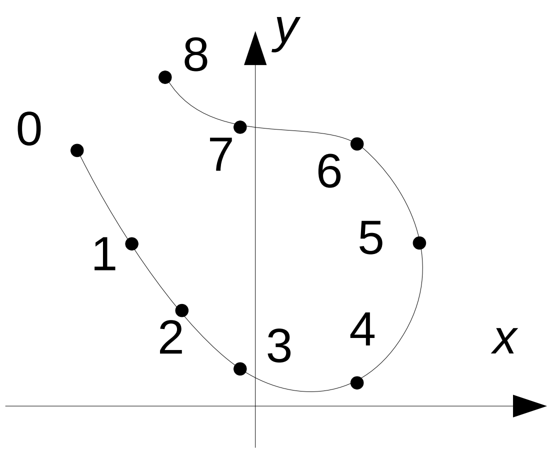
$$\vec{P}(u)=\begin{cases} x=f(u) \\ y=g(u) \\ z=h(u) \end{cases} \quad \text{or} \quad \vec{P}(u,v)=\begin{cases} x=f(u,v) \\ y=g(u,v) \\ z=h(u,v) \end{cases}$$

Parametric curves

Parametric representation

For a curve, the representation takes the following form:

$$\vec{P}(u)=\begin{cases} x=f(u) \\ y=g(u) \\ z=h(u) \end{cases}$$



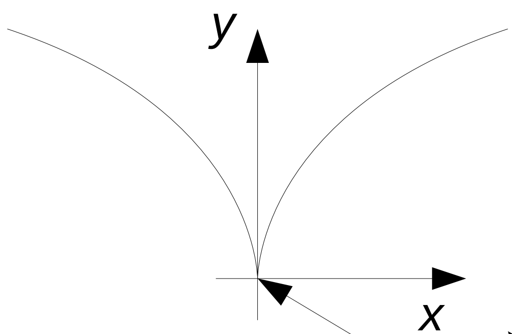
u is a real parameter.

- We obtain all the points of the curve by varying u.
- The parametrization is not unique !
- The parameter u can be limited (limits of the curve)

Continuity of the parametrization

- A parametric curve is said to be C_k if the parametrization $P(u)$ is C_k (i.e. the k^{th} derivative is continuous)
  - However : a parametric curve of class C_∞ may have an angular point... (i.e. some continuity of the parametrization does not necessarily involve the same **geometric** continuity of the resulting curve !)

$$\vec{P}(u)=\begin{cases} x=u^3 \\ y=u^2 \end{cases}$$



$$\vec{P}'(u=0)=\begin{cases} 0 \\ 0 \end{cases}$$

Regularity of the parametrization

- A parametric curve $P(u)$ is *regular* if the first derivative $P'(u)$ does not vanish at any place on all the interval of definition.
- On the contrary, points where $P'(u)$ vanishes, are called *singular* points.
- A given curve can admit two parametric forms such that one is regular and the other one isn't...
  - Example : curve $y=x^2$ with two parametric forms $P(u)$ and $Q(v)$

$$\vec{P}(u)=\begin{cases} x=u \\ y=u^2 \end{cases} \qquad \vec{P}'(u=0)=\begin{cases} 1 \\ 0 \end{cases} \qquad \vec{Q}(v)=\begin{cases} x=v^3 \\ y=v^6 \end{cases} \qquad \vec{Q}'(v=0)=\begin{cases} 0 \\ 0 \end{cases}$$

Regularity and continuity of the parametrization

- A curve of geometric continuity G_k can be described by a parametrization that is non- C_k
- A curve of parametrization C_k may not be G_k
- A curve admitting singular points can be G_k and/or C_k.
- A regular curve is not necessary C_k and/or G_k.

To conclude :

- Parametric continuity, geometric continuity, and regularity are all relatively independent notions!

C_k -equivalent parametric forms

- Two parametric forms $P(u)$ and $Q(v)$ of the same curve are C_k - equivalents if there is a bijective function φ of class C_k whose reciprocal is also of class C_k and such that:

$$v=\varphi(u) \qquad \vec{P}(u)=\vec{Q}(\varphi(u))$$

Length of a regular curve

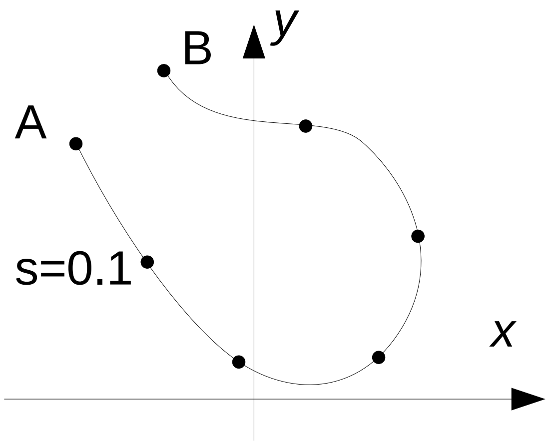
- It is a measure that is independent of the parametrization (provided that those are C_1-equivalent)

$$L=\int_a^b \left|\vec{P}'(u)\right| du=\int_{c=\varphi(a)}^{d=\varphi(b)} \left|\vec{Q}'(v)\right| dv$$

Parametric representation

Natural parametrization

- It is a parametrization s such that :



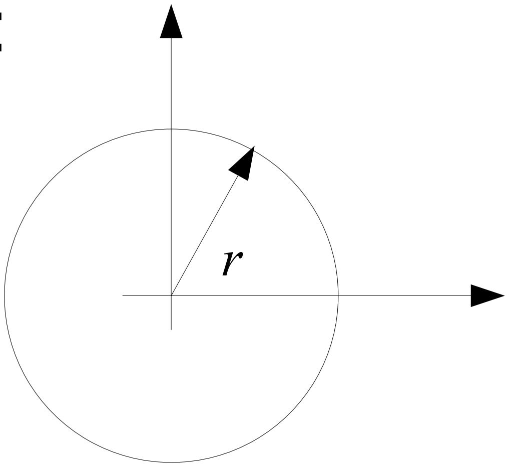
$$L(AB)=s(B)-s(A)$$

- s is also named *curvilinear abscissa along the curve*.
- It can be built from any regular parametrization u of the curve :

$$s(u)=\int_A^u \left|\vec{P}'(t)\right| dt$$

Parametric representation

- Parametrization of a circle in the plane :



$$\begin{cases} x = r\cos u \\ y = r\sin u \end{cases}, \quad u \in [0, 2\pi[$$

- Natural parametrization of the same circle...

$$\begin{cases} x = r\cos \dfrac{s}{r} \\ y = r\sin \dfrac{s}{r} \end{cases}, \quad s \in [0, 2\pi r[\qquad s = r\,u$$

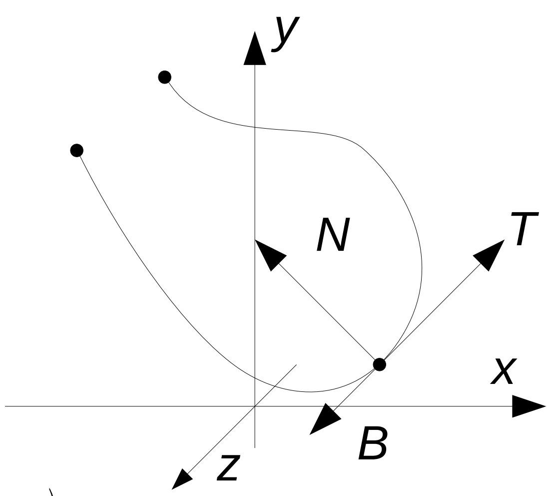
Differential geometry for parametric curves

- Position P :

$$P(u)=\begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} \qquad P'=\frac{dP}{du} \qquad P''=\frac{d^2P}{du^2}$$

- Tangent vector T :

$$T(u)=\frac{dP}{ds}=\frac{dP}{du}\cdot\frac{du}{ds}=\frac{P'}{|P'|}$$



- Normal vector N :

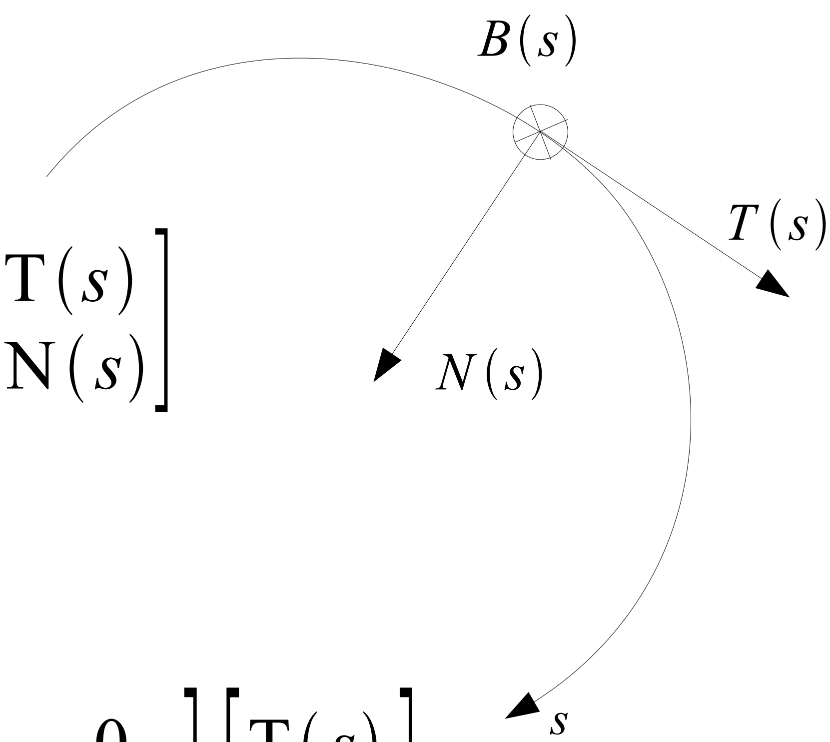
$$N(u)=\frac{Norm(u)}{|Norm(u)|} \text{ with } Norm(u)=P''-(P''\cdot T)T$$

(It is not defined when $Norm(u)=0$!)

- Binormal vector B :

$$B(u)=T(u)\times N(u)$$

Frenet's frame and equations



- 2D $$\begin{bmatrix} \dfrac{d\,\mathrm{T}(s)}{ds} \\ \dfrac{d\,\mathrm{N}(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{T}(s) \\ \mathrm{N}(s) \end{bmatrix}$$
- 3D $$\begin{bmatrix} \dfrac{d\,\mathrm{T}(s)}{ds} \\ \dfrac{d\,\mathrm{N}(s)}{ds} \\ \dfrac{d\,\mathrm{B}(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{T}(s) \\ \mathrm{N}(s) \\ \mathrm{B}(s) \end{bmatrix}$$

Works only with s : the curvilinear abscissa !

Parametric representation

- Curvature κ :

$$|\kappa| = \left|\frac{dT}{ds}\right| = \frac{|P' \times P''|}{|P'|^3}$$

$$\kappa(u) = N(s) \cdot \frac{dT}{ds} = \frac{T'(u) \cdot N(u)}{|P'(u)|}$$

Curvature vector

$$= N(u)\kappa(u)$$

- Torsion τ:

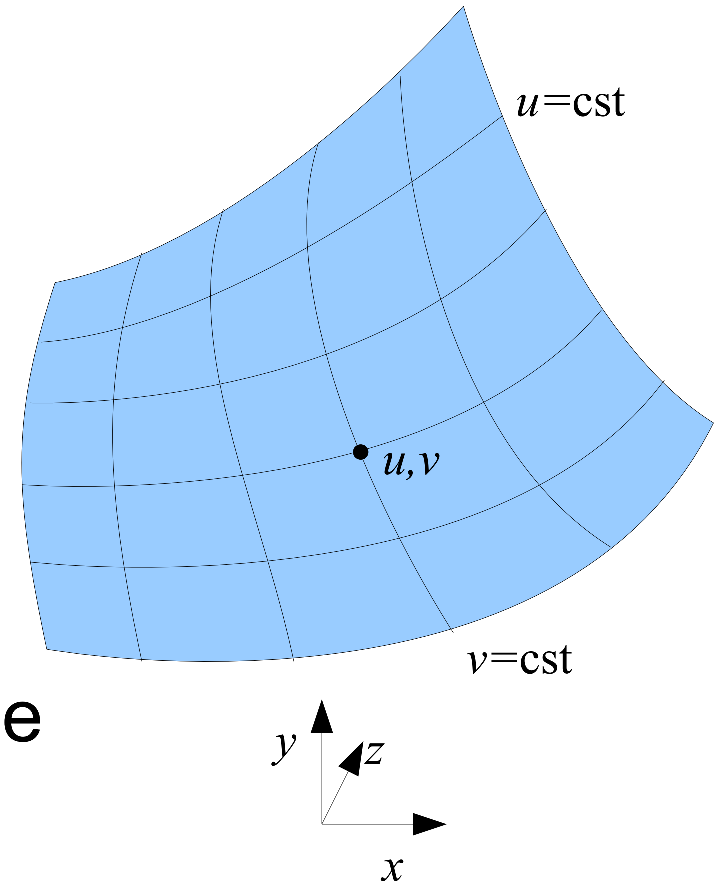
$$\tau(u) = \frac{N'(u) \cdot B(u)}{|P'(u)|} \qquad \tau = \frac{\left(\frac{d^2T}{ds^2}, \frac{dT}{ds}, T\right)}{\kappa^2} = \frac{(P''', P'', P')}{|P' \times P''|^2}$$

Parametric surfaces

Parametric representation

A parametric surface is represented as :

$$\vec{P}(u,v)=\begin{cases} x=f(u,v) \\ y=g(u,v) \\ z=h(u,v) \end{cases}$$



u,v are two real parameters

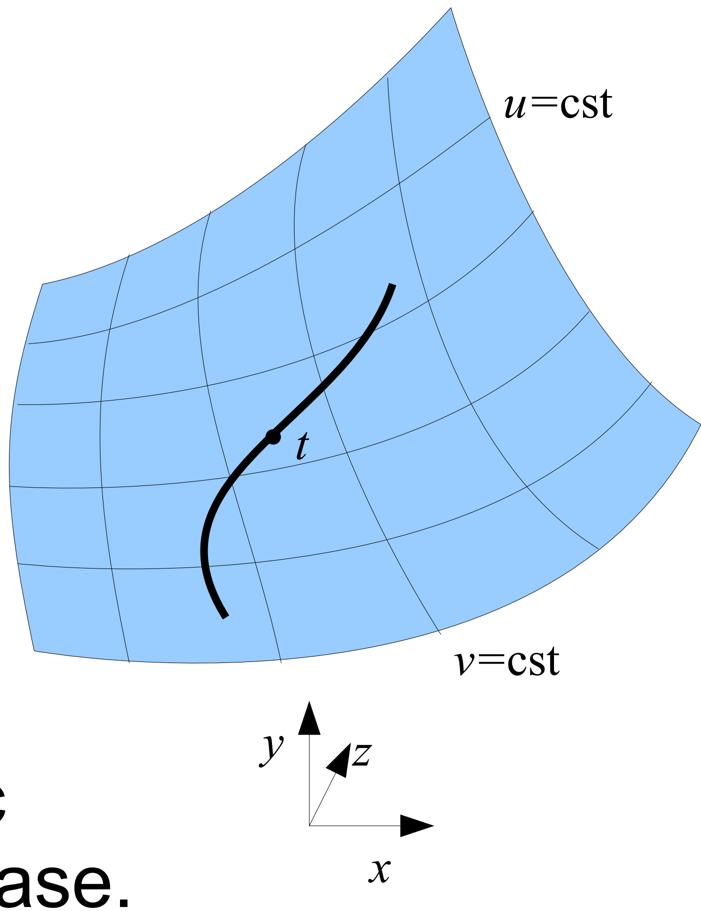
- All the points of the surface can be obtained by varying u and v.
- Again, the parametrization is not unique !
- The interval of definition is limited (limits the surface)

Parametric representation

A curve can be defined on the surface

Parametric space « Ambient » space

$$\vec{\Gamma}^{uv}(t):\begin{cases} u=u(t) \\ v=v(t) \end{cases} \longrightarrow \vec{P}(u,v):\begin{cases} x=f(u,v) \\ y=g(u,v) \\ z=h(u,v) \end{cases}$$



$$\vec{\Gamma}(t):\begin{cases} x=f(u(t),v(t)) \\ y=g(u(t),v(t)) \\ z=h(u(t),v(t)) \end{cases}$$

- The equations seen for parametric curves are valid in this particular case.

Regularity and continuity of the parametrization

- A parametric surface is of class C_k if the mapping $P(u,v)$ onto R^3 is of class C_k.
- A parametrization is *regular* if and only if (iff)

$$\frac{\partial \vec{P}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{P}}{\partial v}(u_0, v_0) \neq \vec{0} \quad \forall (u_0, v_0) \in D \subset \mathbb{R}^2$$

- The points on the parametric space for which it is not the case are called *singular* points.
- C_k–equivalent parametric forms follow the same definitions as for curves

Parametric representation

Differential geometry for parametric surfaces

- Position P : $\vec{P}(u,v)=\begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ $\quad \vec{P}^{\mathrm{u}}=\dfrac{\partial \vec{P}}{\partial u} \quad \vec{P}^{\mathrm{uv}}=\dfrac{\partial^2 \vec{P}}{\partial u\, \partial v} \quad \cdots$
- Unit tangent vectors T^{u} and T^{v} :

$$\vec{T}^{\mathrm{u}}(u,v)=\frac{\partial P}{\partial u}\cdot\left|\frac{\partial P}{\partial u}\right|^{-1}=\frac{P^{\mathrm{u}}}{\left|P^{\mathrm{u}}\right|} \qquad \vec{T}^{\mathrm{v}}(u,v)=\frac{\partial P}{\partial v}\cdot\left|\frac{\partial P}{\partial v}\right|^{-1}=\frac{P^{\mathrm{v}}}{\left|P^{\mathrm{v}}\right|}$$

  - These vectors are, in general, not orthogonal !
- Tangent plane (parametric form)

$$\vec{Pt}_{(u_0,v_0)}(a,b)=\vec{P}(u_0,v_0)+a\cdot\vec{T}^{\mathrm{u}}(u_0,v_0)+b\cdot\vec{T}^{\mathrm{v}}(u_0,v_0)$$

or

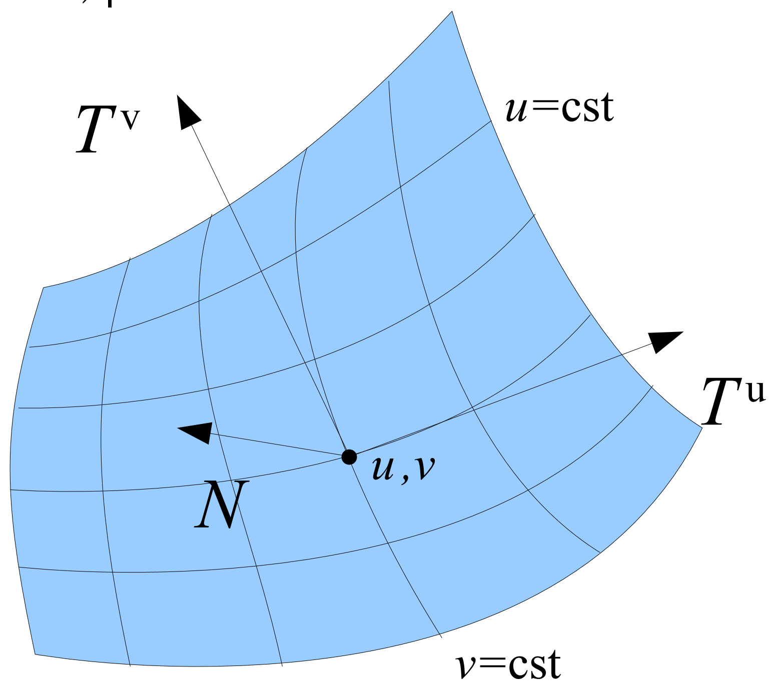
$$(a,b)\in\mathbb{R}^2$$

$$\vec{Pt}_{(u_0,v_0)}(a,b)=\vec{P}(u_0,v_0)+a\cdot\vec{P}^{\mathrm{u}}(u_0,v_0)+b\cdot\vec{P}^{\mathrm{v}}(u_0,v_0)$$

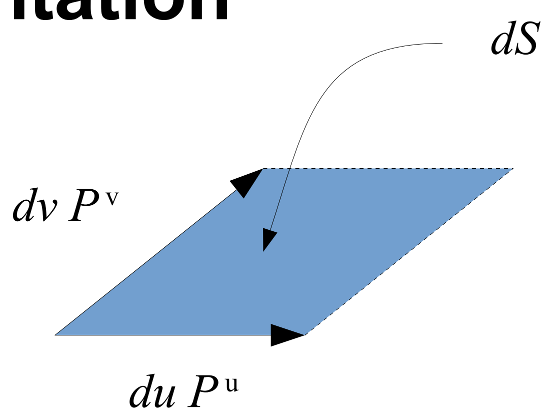
Parametric representation

- Normal vector N :

$$N(u,v)=\frac{Norm(u,v)}{|Norm(u,v)|} \text{ with } Norm(u,v)=T^{\mathrm{u}}\times T^{\mathrm{v}} \text{ or } P^{\mathrm{u}}\times P^{\mathrm{v}}$$



Parametric representation



- Area of a surface

$$A=\iint_{\Omega} dS$$

$$dS=\left|du\cdot P^{u}\times dv\cdot P^{v}\right|=\left|P^{u}\times P^{v}\right|dudv$$

1st fundamental form

- Other notation of the area of a surface

$$|\vec{a}\times\vec{b}|^{2}=(\vec{a}\cdot\vec{a})\cdot(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})^{2} \longleftarrow \boxed{\text{Lagrange's identity}}$$

$$dS=\sqrt{(e\,g-f^{2})}\,dudv \quad \text{with } e=P^{u}\cdot P^{u}\ ,\ f=P^{u}\cdot P^{v}\ ,\ g=P^{v}\cdot P^{v}$$

$$A=\iint_{D}\sqrt{(eg-f^{2})}\,dudv$$

Parametric representation

- Length of a curve on a surface

$$\vec{P}(u,v):\begin{cases} x=f(u,v) \\ y=g(u,v) \\ z=h(u,v) \end{cases} \qquad P^{u}=\frac{\partial P(u,v)}{\partial u} \qquad P^{v}=\frac{\partial P(u,v)}{\partial v}$$

$$\vec{\Gamma}^{uv}(t):\begin{cases} u=u(t) \\ v=v(t) \end{cases} \qquad \vec{\Gamma}(t):\begin{cases} x=f(u(t),v(t)) \\ y=g(u(t),v(t)) \\ z=h(u(t),v(t)) \end{cases} \qquad \Gamma'=\frac{dP(u(t),v(t))}{dt}$$

$$\Gamma'=\frac{\partial P(u(t),v(t))}{\partial u}\frac{du}{dt}+\frac{\partial P(u(t),v(t))}{\partial v}\frac{dv}{dt}$$

Derivative of the curve expressed in the parametric space of the surface

Derivative of the surface (in 3D)

$$\Gamma'(t)=u'(t)P^{u}(u(t),v(t))+v'(t)P^{v}(u(t),v(t))$$

Parametric representation

$$L=\int_a^b \left|\vec{\Gamma}'(t)\right| dt=\int_a^b \sqrt{\left|\vec{\Gamma}'(t)\right|^2}\, dt$$

$$\Gamma'(t)=u'(t)\,P^{\mathrm{u}}(u(t),v(t))+v'(t)\,P^{\mathrm{v}}(u(t),v(t))$$

$$|\Gamma'(t)|^2=e\,u'(t)^2+2\,f\,u'(t)\,v'(t)+g\,v'(t)^2$$

with $e=P^u\cdot P^u$, $f=P^u\cdot P^v$, $g=P^v\cdot P^v$

- If we set

$$ds=\sqrt{e\,u'(t)^2+2f\,u'(t)v'(t)+g\,v'(t)^2}\,dt$$

that is equivalent to : $ds=\sqrt{e\,du^2+2f\,du dv+g\,dv^2}$

(we have $L=\int_{s(a)}^{s(b)} ds=s(b)-s(a)$)

, we get a quadratic form :

$$e\,u'(t)^2+2f\,u'(t)v'(t)+g\,v'(t)^2=\begin{pmatrix}u'(t) & v'(t)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u'(t)\\ v'(t)\end{pmatrix}$$

$$L=\int_a^b \sqrt{\begin{pmatrix}u'(t) & v'(t)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u'(t)\\ v'(t)\end{pmatrix}}\,dt$$

Parametric representation

- Angle between two curves ...

$$\Gamma_1'(t_1)\cdot\Gamma_2'(t_2)=|\Gamma_1'(t_1)||\Gamma_2'(t_2)|\cos\alpha=\begin{pmatrix}u_1'(t_1) & v_1'(t_1)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u_2'(t_2)\\ v_2'(t_2)\end{pmatrix}$$

$$\cos\alpha=\frac{\begin{pmatrix}u_1'(t_1) & v_1'(t_1)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u_2'(t_2)\\ v_2'(t_2)\end{pmatrix}}{\sqrt{\begin{pmatrix}u_1'(t_1) & v_1'(t_1)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u_1'(t_1)\\ v_1'(t_1)\end{pmatrix}\begin{pmatrix}u_2'(t_2) & v_2'(t_2)\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}u_2'(t_2)\\ v_2'(t_2)\end{pmatrix}}}$$

Parametric representation

- The first fundamental form is the application :

$$\varphi_1\left(d\,\Gamma_1^{uv},d\,\Gamma_2^{uv}\right)=\begin{pmatrix}du_1 & dv_1\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}\begin{pmatrix}du_2\\ dv_2\end{pmatrix}=\begin{pmatrix}du_1 & dv_1\end{pmatrix}\mathrm{M}_1\begin{pmatrix}du_2\\ dv_2\end{pmatrix}$$

with $e=P^u\cdot P^u$, $f=P^u\cdot P^v$, $g=P^v\cdot P^v$

  - It is a symmetric bilinear form that allows to measure real distances from variations in the parametric space...
  - The matrix M_1 is a representation of the metric tensor, often noted $g_{ij}, i,j=u,v$
  - M_1 also is related to the Jacobian matrix of the transformation $(u,v)\rightarrow(x,y,z)$ (it is J^TJ).

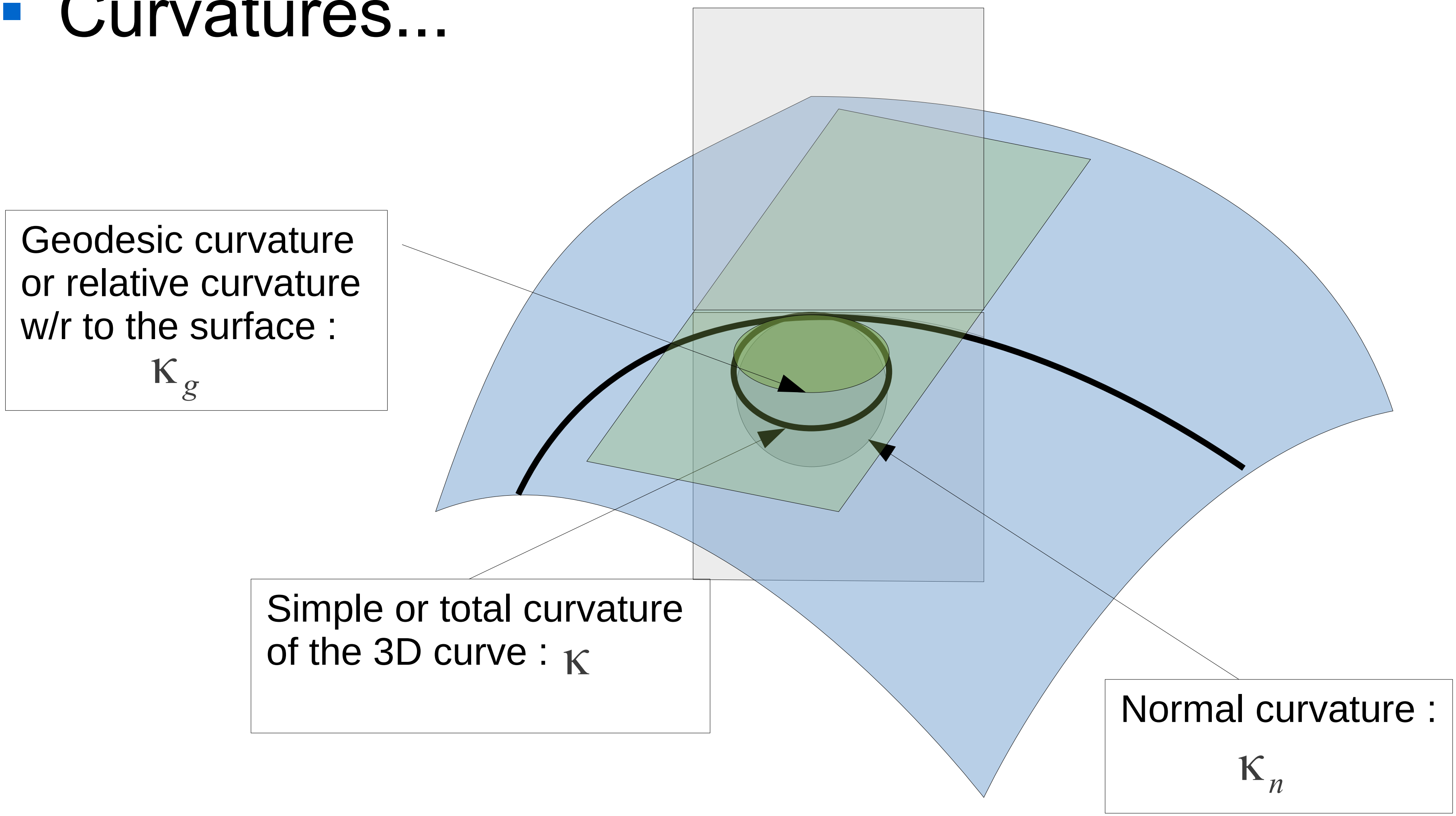
$$J=\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\\ \partial z/\partial u & \partial z/\partial v\end{pmatrix}$$

$$L=\int_a^b\sqrt{\varphi\left(d\,\Gamma^{uv},d\,\Gamma^{uv}\right)}\,dt \qquad \cos\alpha=\frac{\varphi\left(d\,\Gamma_1^{uv},d\,\Gamma_2^{uv}\right)}{\sqrt{\varphi\left(d\,\Gamma_1^{uv},d\,\Gamma_1^{uv}\right)\varphi\left(d\,\Gamma_2^{uv},d\,\Gamma_2^{uv}\right)}}$$

$$A=\iint_D\sqrt{\det \mathrm{M}_1}\,du\,dv \quad \left(=\iint_D \det \mathrm{J}\,du\,dv \text{ under some conditions}\right)$$

Parametric representation

- Curvatures...



Parametric representation

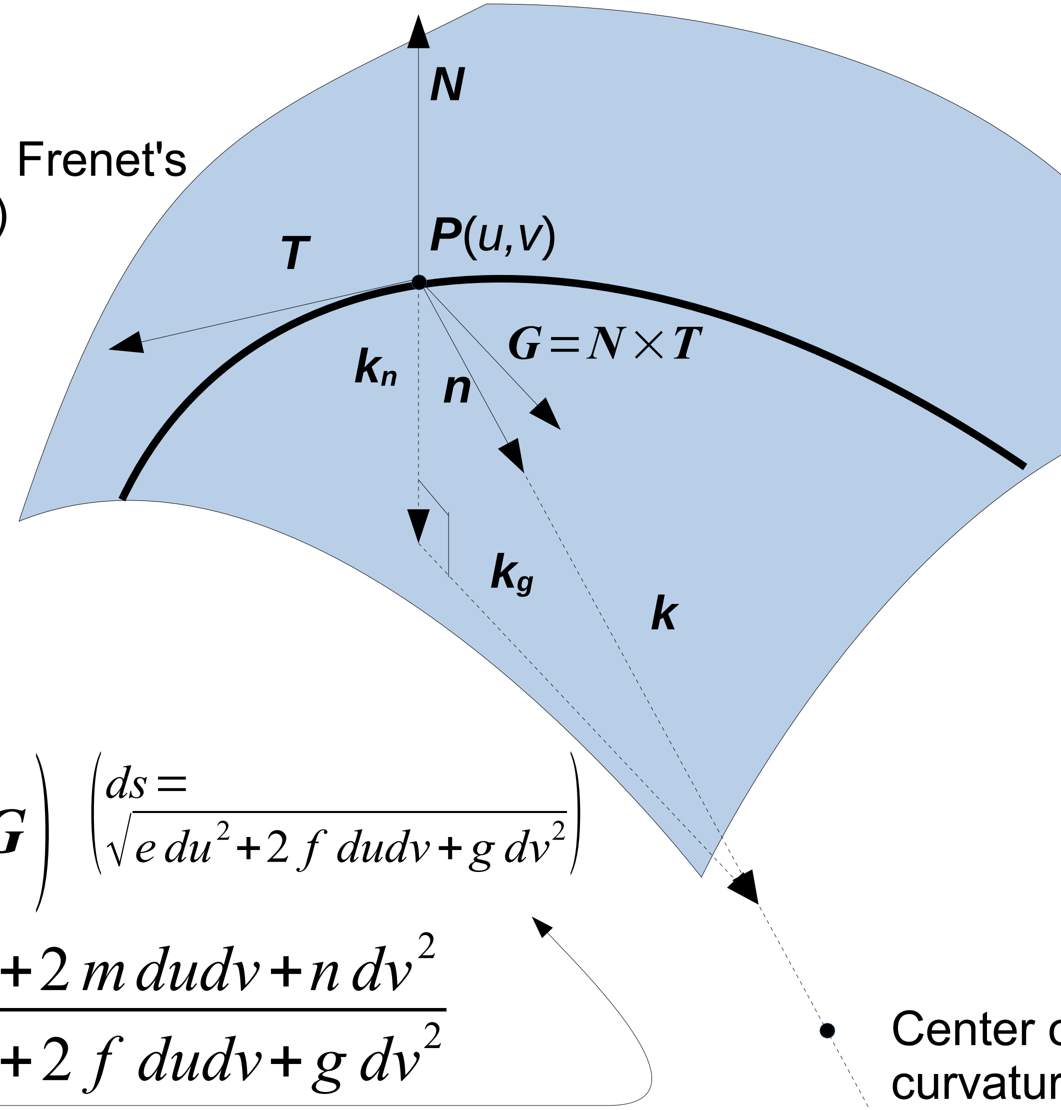
- Curvature vectors

$$\boldsymbol{k}=\frac{d\,\boldsymbol{T}}{ds}=\kappa\cdot\boldsymbol{n}=\boldsymbol{k}_n+\boldsymbol{k}_g$$ (e.g. from Frenet's relations)

$$\boldsymbol{k}_n=\kappa_n\cdot\boldsymbol{N}\qquad \boldsymbol{k}_g=\kappa_g\cdot\boldsymbol{G}$$

  - Necessarily, we do have : $\boldsymbol{N}\cdot\boldsymbol{T}=0$
  - Differentiating yields :

$$\frac{d\,\boldsymbol{T}}{ds}\cdot\boldsymbol{N}+\boldsymbol{T}\cdot\frac{d\,\boldsymbol{N}}{ds}=0$$



$$\kappa_n=\frac{d\,\boldsymbol{T}}{ds}\cdot\boldsymbol{N}=-\boldsymbol{T}\cdot\frac{d\,\boldsymbol{N}}{ds}\qquad\left(\kappa_g=\frac{d\,\boldsymbol{T}}{ds}\cdot\boldsymbol{G}\right)\qquad\left(ds=\sqrt{e\,du^2+2\,f\,du dv+g\,dv^2}\right)$$

$$=-\frac{d\,\boldsymbol{P}(u,v)}{ds}\cdot\frac{d\,\boldsymbol{N}}{ds}=-\frac{d\,\boldsymbol{P}\cdot d\,\boldsymbol{N}}{ds^2}=\frac{l\,du^2+2\,m\,du dv+n\,dv^2}{e\,du^2+2\,f\,du dv+g\,dv^2}$$

with

$$l=-\frac{dP}{du}\cdot\frac{d\,\boldsymbol{N}}{du}\qquad m=-\frac{1}{2}\left(\frac{dP}{du}\cdot\frac{d\,\boldsymbol{N}}{dv}+\frac{dP}{dv}\cdot\frac{d\,\boldsymbol{N}}{du}\right)\qquad n=-\frac{dP}{dv}\cdot\frac{d\,\boldsymbol{N}}{dv}$$

Parametric representation

$$l = -\frac{dP}{du} \cdot \frac{d\,\mathbf{N}}{du} \qquad m = -\frac{1}{2}\left(\frac{dP}{du} \cdot \frac{d\,\mathbf{N}}{dv} + \frac{dP}{dv} \cdot \frac{d\,\mathbf{N}}{du}\right) \qquad n = -\frac{dP}{dv} \cdot \frac{d\,\mathbf{N}}{dv}$$

- This can be rewritten by noting that $\frac{dP}{du} \cdot N = 0$ and $\frac{dP}{dv} \cdot N = 0$

  , and integrating by parts $UV = \int U\,dV + \int V\,dU$

  where *e.g.* for l : $U = \frac{dP}{du} \qquad dV = \frac{dN}{du} \cdot du$

$$l = \frac{d^2 P}{du^2} \cdot \mathbf{N} \qquad m = \frac{d^2 P}{du\,dv} \cdot \mathbf{N} \qquad n = \frac{d^2 P}{dv^2} \cdot \mathbf{N}$$

Curvatures and the second fundamental form

- The second fundamental form is the application

$$\varphi_2\left(d\,\Gamma_1^{uv}, d\,\Gamma_2^{uv}\right)=\begin{pmatrix} du_1 & dv_1 \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix}\begin{pmatrix} du_2 \\ dv_2 \end{pmatrix}=\begin{pmatrix} du_1 & dv_1 \end{pmatrix}\mathrm{M}_2\begin{pmatrix} du_2 \\ dv_2 \end{pmatrix}$$

$$\text{with} \quad l=N\cdot P^{\mathrm{uu}} \;,\; n=N\cdot P^{\mathrm{vv}} \;,\; m=N\cdot P^{\mathrm{uv}}$$

- Normal curvature of a curve on a surface :

$$\kappa_n=\frac{\varphi_2\left(d\,\Gamma^{uv}, d\,\Gamma^{uv}\right)}{ds^2} \qquad \kappa_n=\frac{\varphi_2\left(d\,\Gamma^{uv}, d\,\Gamma^{uv}\right)}{\varphi_1\left(d\,\Gamma^{uv}, d\,\Gamma^{uv}\right)}$$

- Geodesic curvature (curvature in the tangent plane)
  - Measures the deviation of the curve in comparison with a geodesic

Definition of the geodesic

- It is a curve with a minimal length on a surface.
  - In the plane, those are lines
  - On a sphere, great circles
  - By definition, the geodesic has no geodesic curvature.

  Relations between « simple » curvature, geodesic curvature and normal curvature is obtained from the curvature vectors

  $$\boldsymbol{k}=\boldsymbol{k_n}+\boldsymbol{k_g} \quad \boldsymbol{k_n}=\kappa_n\cdot N \quad \boldsymbol{k}=\kappa\cdot\boldsymbol{n} \quad \boldsymbol{k_g}=\kappa_g\cdot(N\times\boldsymbol{T})=\kappa_g\cdot\boldsymbol{G}$$

  $$\kappa^2=\kappa_n^2+\kappa_g^2$$

  The geodesic curvature is therefore readily obtained from the others (except for the sign, which is same as $\frac{d\,\boldsymbol{T}}{dt}\cdot\boldsymbol{G}$)

  $$\kappa_g^2=\kappa^2-\kappa_n^2$$

Parametric representation

- A circle of radius R in the plane

  Curvature $\kappa = 1/R$

  Geodesic curvature $\kappa_g = 1/R$

  Normal curvature $\kappa_n = 0$

- A equatorial circle on a sphere of radius R

  Curvature $\kappa = 1/R$

  Geodesic curvature $\kappa_g = 0$

  Normal curvature $\kappa_n = 1/R$

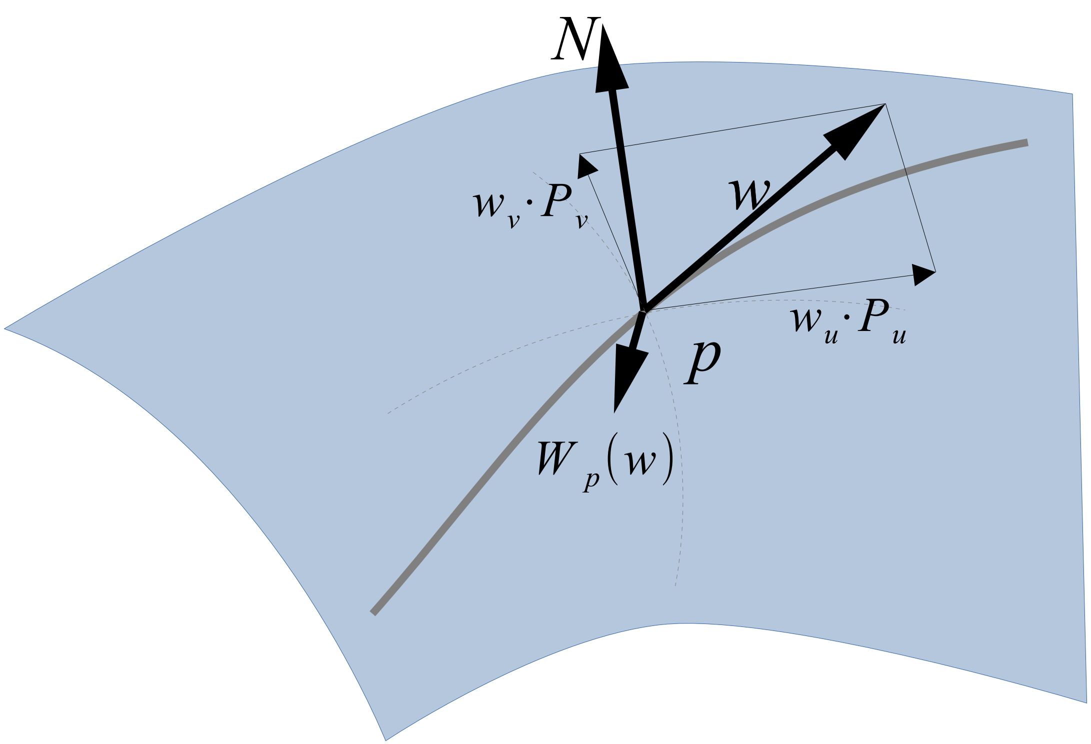
- In the previous slides, κ is the curvature of the curve in 3D (not in the parametric space !!!)

Parametric representation

- Curvatures of the surface $\kappa_n = \dfrac{\varphi_2\left(d\,\Gamma^{uv}, d\,\Gamma^{uv}\right)}{\varphi_1\left(d\,\Gamma^{uv}, d\,\Gamma^{uv}\right)}$
  - The normal curvature κ_n at a point only depends on the local orientation of Γ on the surface. If we sweep through all the orientations, this curvature passes by a minimum κ_{min} then a maximum κ_{max} for two perpendicular directions.
  - These directions and the particular values of κ_n are respectively called **principal directions** and **principal curvatures** of the surface. They do not depend on the parametrization of the surface...
  - They are the solution of an eigenvalue problem.
  - The **Shape operator** (also called **Weingarten Map**) plays a key role here, see what follows.

Parametric representation

- The shape operator $W_p(w) = -\nabla_w \cdot N$ $\quad w = w_u \cdot P_u + w_v \cdot P_v$
  - Here, N is the normal to the surface, p is a point on the surface, w is a tangent unit vector to the surface at p. ∇_w is a directional derivative along w (in the surface) :
    it is equal to $w_u \frac{\partial}{\partial u} + w_v \frac{\partial}{\partial v}$.
    This operator gives back a vector in the tangent plane.



Parametric representation

- The shape operator $W_p(w) = -\nabla_w \cdot N$
  - If one uses an orthonormal basis, eigenvectors are easy to compute as the operator is **symmetric**...
  - Let's build such an orthonormal basis from the natural one :

$$P^{\mathrm{u}} = \frac{\partial P}{\partial u} \qquad P^{\mathrm{v}} = \frac{\partial P}{\partial v} \qquad N = N(u, v) = \frac{P^{\mathrm{u}} \times P^{\mathrm{v}}}{\left\| P^{\mathrm{u}} \times P^{\mathrm{v}} \right\|}$$

$$\longrightarrow \quad t_1 = \frac{P^{\mathrm{u}}}{\left\| P^{\mathrm{u}} \right\|} \qquad t_2 = N \times t_1 = \frac{P^{\mathrm{v}} - (P^{\mathrm{v}} \cdot t_1) t_1}{\left\| P^{\mathrm{v}} - (P^{\mathrm{v}} \cdot t_1) t_1 \right\|}$$

Parametric representation

- It is then easy to show that one can switch from one basis to the other using coefficients of the 1st fundamental form …

$$t_1 = \frac{P^{\mathrm{u}}}{\|P^{\mathrm{u}}\|} = \frac{P^{\mathrm{u}}}{\sqrt{e}} \qquad t_2 = \frac{e\,P^{\mathrm{v}} - f\,P^{\mathrm{u}}}{\sqrt{e}\sqrt{eg-f^2}} = \frac{e\,P^{\mathrm{v}} - f\,P^{\mathrm{u}}}{\sqrt{e}\sqrt{h}}$$

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \frac{1}{\sqrt{e}\sqrt{h}} \begin{bmatrix} \sqrt{h} & 0 \\ -f & e \end{bmatrix} \begin{bmatrix} P^{\mathrm{u}} \\ P^{\mathrm{v}} \end{bmatrix} \qquad h = eg - f^2 = det(M_1)$$

- One can also switch the other way back, working on coefficients rather than on vectors e.g. for any vector $\vec{w} = w_1 t_1 + w_2 t_2 = w_u P^{\mathrm{u}} + w_v P^{\mathrm{v}}$

One gets $$\begin{bmatrix} w_u \\ w_v \end{bmatrix} = \frac{1}{\sqrt{e}\sqrt{h}} \begin{bmatrix} \sqrt{h} & -f \\ 0 & e \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Parametric representation

- Now back to the shape operator : we will start with a vector expressed in the orthonormal frame (w_1,w_2) , convert it to the parametric basis (w_u,w_v), take the directional derivative $w_u \frac{\partial}{\partial u}+w_v \frac{\partial}{\partial v}$ of the normal N and then project back to the orthonormal frame.

$$W_p(w)=-\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}\left[n_u \quad n_v\right]\Phi\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$=-\Phi^T\begin{bmatrix} P^{\mathrm{u}} \\ P^{\mathrm{v}} \end{bmatrix}\left[n_u \quad n_v\right]\Phi\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- Here, $-\begin{bmatrix} P^{\mathrm{u}} \\ P^{\mathrm{v}} \end{bmatrix}\left[n_u \quad n_v\right]$ is precisely the matrix of the coefficients of the second fundamental form...

Parametric representation

- In the orthonormal basis

$$W_p(w) = \Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \Phi \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{eh} \begin{bmatrix} \sqrt{h} & 0 \\ -f & e \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} \sqrt{h} & -f \\ 0 & e \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \frac{1}{eh} \begin{bmatrix} hl & (em - fl)\sqrt{h} \\ (em - fl)\sqrt{h} & f^2 l + e^2 n - 2efm \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- In the original basis

$$\tilde{W}_p(w) = \Phi W_p(w) \Phi^{-1} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \Phi \left(\Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \Phi \right) \Phi^{-1} \begin{bmatrix} w_u \\ w_v \end{bmatrix}$$

$$= \Phi \Phi^T \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \frac{1}{h} \begin{bmatrix} g & -f \\ -f & e \end{bmatrix} \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix}^{-1} \cdot \begin{bmatrix} l & m \\ m & n \end{bmatrix} \begin{bmatrix} w_u \\ w_v \end{bmatrix}$$

(Weingarten Equation)

Parametric representation

- Eigenvalues and eigenvectors of the shape operator
  - The shape operator tells the behavior of the normal for a given orientation (*i.e.* the derivative of the unit normal vector with respect to a small increment along that orientation on the surface)
  - One can work either in the original frame (non orthonormal):

$$(L-\kappa I)\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ with } L=M_1^{-1}M_2=\widetilde{W}_p(w) \qquad det(L-\kappa I)=0$$

… or the orthonormal frame, for which the matrix is symmetric and the 2D eigenvectors are orthogonal

$$(W_p(w)-\kappa I)\begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad det(W_p(w)-\kappa I)=0$$

Parametric representation

$$(M_2-\kappa M_1)\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}l-\kappa e & m-\kappa f\\ m-\kappa f & n-\kappa g\end{pmatrix}\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

- The eigenvectors are obtained using any of the relations:

$$\begin{pmatrix}du\\dv\end{pmatrix}\cdot\begin{pmatrix}l-\kappa e\\ m-\kappa f\end{pmatrix}=0 \longrightarrow \begin{pmatrix}du\\dv\end{pmatrix}=\alpha\cdot\begin{pmatrix}m-\kappa f\\ \kappa e-l\end{pmatrix} \;,\quad \alpha>0: du^2+dv^2=1$$

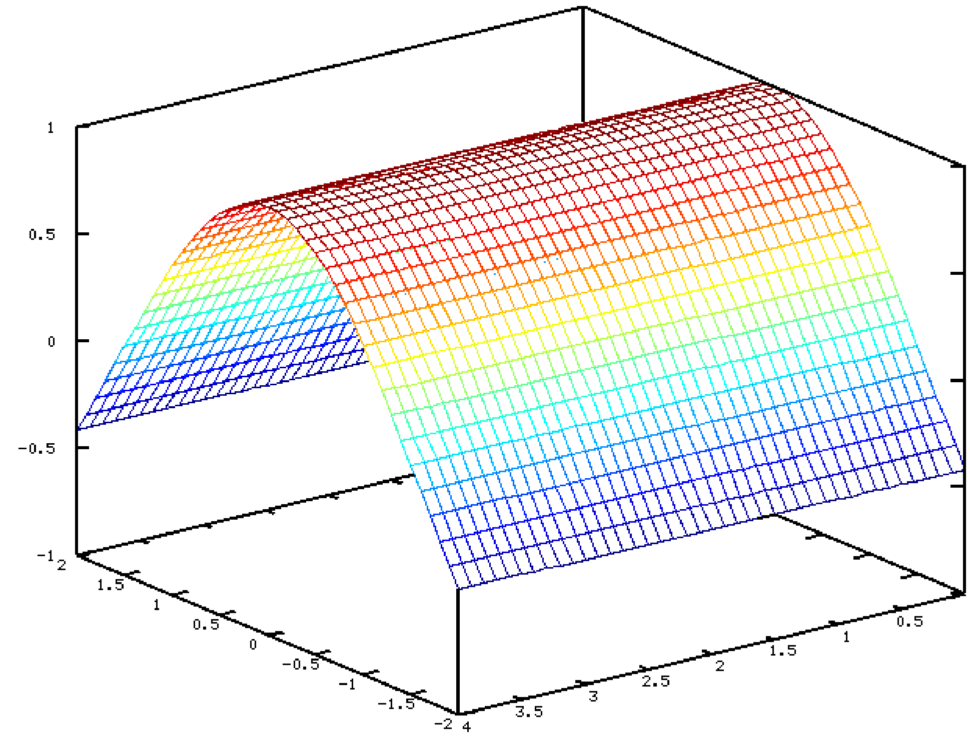
- Just use the other relation if the basis contains a null vector for one of the values of κ :

$$\begin{pmatrix}du\\dv\end{pmatrix}\cdot\begin{pmatrix}m-\kappa f\\ l-\kappa g\end{pmatrix}=0 \longrightarrow \begin{pmatrix}du\\dv\end{pmatrix}=\alpha\cdot\begin{pmatrix}l-\kappa g\\ \kappa f-m\end{pmatrix} \;,\quad \alpha>0: du^2+dv^2=1$$

Parametric representation

- Example

$$P(u,v)=\begin{pmatrix} u \\ v \\ \cos u \end{pmatrix}$$



$$P^u(u,v)=\begin{pmatrix} 1 \\ 0 \\ -\sin u \end{pmatrix} \qquad P^v(u,v)=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$P^{uu}(u,v)=\begin{pmatrix} 0 \\ 0 \\ -\cos u \end{pmatrix} \qquad P^{uv}(u,v)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P^{vv}(u,v)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad N(u,v)=\frac{P^u\times P^v}{\left|P^u\times P^v\right|}=\begin{pmatrix} -\sin u \\ 0 \\ 1 \end{pmatrix}\cdot\frac{1}{\sqrt{1+\sin^2 u}}$$

Parametric representation

$$(L-\kappa I)\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ with } L=M_1^{-1}M_2$$

$$\mathrm{M}_1=\begin{pmatrix} P^u\cdot P^u & P^u\cdot P^v \\ P^u\cdot P^v & P^v\cdot P^v \end{pmatrix}=\begin{pmatrix} 1+\sin^2 u & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{M}_2=\begin{pmatrix} N\cdot P^{uu} & N\cdot P^{uv} \\ N\cdot P^{uv} & N\cdot P^{vv} \end{pmatrix}=\begin{pmatrix} \dfrac{-\cos u}{\sqrt{1+\sin^2 u}} & 0 \\ 0 & 0 \end{pmatrix}$$

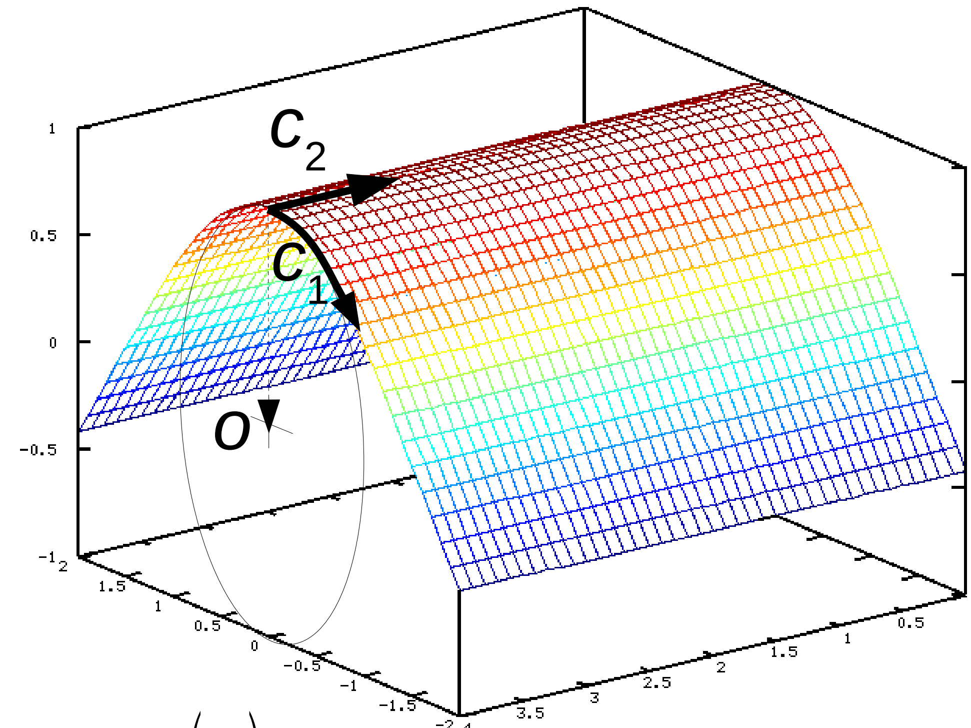
Parametric representation

- Numerical application : u=0 and v=0

$$M_1=\begin{pmatrix}1+\sin^2 u & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$M_1^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$M_2=\begin{pmatrix}\dfrac{-\cos u}{\sqrt{1+\sin^2 u}} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}$$



$$L=M_1^{-1}M_2=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\longrightarrow$$

$$\kappa_1=-1 \qquad c_1=\begin{pmatrix}1\\0\end{pmatrix} \qquad \kappa_2=0 \qquad c_2=\begin{pmatrix}0\\1\end{pmatrix}\longrightarrow$$

$$C_1^{3D}=(P_u,P_v)\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

$$C_2^{3D}=(P_u,P_v)\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

Umbilical points

- Points whose normal curvature κ_n is independent of the orientation of Γ .
  - Locally, the surface looks like a plane or a sphere.
- In this case, the following relations hold for the matrices M_1 and M_2 :

$$\frac{l}{e}=\frac{m}{f}=\frac{n}{g} \text{ if } f\neq 0$$

$$\frac{l}{e}=\frac{n}{g} \text{ if } f=0 \text{ and } m=0$$

$$\kappa_n=\frac{l}{e}=\kappa_{min}=\kappa_{max}=\text{ const.}$$

Principal directions are **arbitrary**.

- Gaussian curvature and average curvature
  - Invariants (do not depend on the parametrization)

$$\kappa_T = \kappa_{min}\kappa_{max} = \det(L) = \frac{\det(M_2)}{\det(M_1)} = \frac{l\,n - m^2}{e\,g - f^2}$$

$$\kappa_M = \frac{\kappa_{min} + \kappa_{max}}{2} = \frac{1}{2}\,\mathrm{trace}(L) = \frac{1}{2}\,\frac{e\,n + g\,l - 2\,f\,m}{e\,g - f^2}$$

$$\left(L = M_1^{-1} M_2\right)$$

Local nature of the surface

- $\kappa_T > 0$ the surface is locally an ellipsoid – elliptic point
- $\kappa_T < 0$: hyperbolic paraboloid – saddle point – hyperbolic point
- $\kappa_T = 0$ and $\kappa_M <> 0$: parabolic cylinder – parabolic point
- $\kappa_T = 0$ and $\kappa_M = 0$: no information – locally plane

Christoffel symbols

- Complete developments. Christoffel symbols
  - Expression of the first derivatives along a curve...

$$P_u = \frac{\partial P(u,v)}{\partial u} \quad u' = \frac{du(t)}{dt}$$

$$P_v = \frac{\partial P(u,v)}{\partial v} \quad v' = \frac{dv(t)}{dt}$$

$$\Gamma'(t) = u' P_u + v' P_v$$

  - Lets differentiate that expression again...

Christoffel symbols

- $\left(\Gamma'(t) = u' P_u + v' P_v\right)'$

$$\Gamma''(t) = u'' P_u + u'\left(u' P_{uu} + v' P_{uv}\right) + v'' P_v + v'\left(u' P_{vu} + v' P_{vv}\right)$$
$$= u'' P_u + v'' P_v + u'^2 P_{uu} + u' v' P_{uv} + u' v' P_{vu} + v'^2 P_{vv}$$

  - The terms $u'' P_u + v'' P_v$ are in the tangent plane
  - The terms $P_{ij}, i, j = \{u, v\}$ can be expressed as linear combinations of tangential and normal components :

$$P_{ij} = C^u_{ij} P_u + C^v_{ij} P_v + L_{ij} N \longleftarrow \text{Gauss formula}$$

The C^k_{ij} are the Christoffel symbols (often written Γ^k_{ij} in textbooks) and the L_{ij} are the coefficients of the second fundamental form.

Christoffel symbols

- Using Einstein's notation (summation over repeated indices): $P_{ij} = C_{ij}^k P_k + L_{ij} N\,,\quad i, j, k = u, v$
  - How to compute C_{ij}^k and L_{ij}?

    $\rightarrow$ Multiply by $P_l\,,\quad l = u, v$

$$P_{ij} \cdot P_l = C_{ij}^k P_k \cdot P_l + L_{ij} \underbrace{N \cdot P_l}_{0}$$

Coefficients of the 1st fundamental form $= g_{kl} =$ metric tensor

$$P_{ij} \cdot P_l \cdot (g_{kl})^{-1} = C_{ij}^k g_{kl} \cdot g_{kl}^{-1}$$

$$C_{ij}^k = P_{ij} \cdot P_l \cdot (g_{kl})^{-1} \longrightarrow C_{ij}^k = P_{ij} \cdot P_l \cdot g^{lk}$$

Conventional notation of the inverse of the metric tensor

Christoffel symbols

$$P_{ij} = C_{ij}^{k} P_k + L_{ij} N$$

- Now, multiply by N

$$P_{ij} \cdot N = C_{ij}^{k} \underbrace{P_k \cdot N}_{0} + L_{ij} \underbrace{N \cdot N}_{1}$$

$$L_{ij} = P_{ij} \cdot N$$

- One recovers the formula seen before:

$$L_{ij} \equiv M_2 \qquad \mathrm{M}_2 = \begin{pmatrix} N \cdot P^{uu} & N \cdot P^{uv} \\ N \cdot P^{uv} & N \cdot P^{vv} \end{pmatrix}$$

Christoffel symbols

- Use of Christoffel symbols and the second fundamental form
  - Let us return to the expression

$$\Gamma''(t)=u''P_u+v''P_v+u'^2P_{uu}+u'v'P_{uv}+u'v'P_{vu}+v'^2P_{vv}$$

  - Now replace u, v by $u^i, i=1,2$ and use the Einstein notation again :

$$\Gamma''(t)=(u^i)''P_i+(u^i)'\cdot(u^j)'P_{ij}$$

  - Now substitute $P_{ij}=C_{ij}^k P_k+L_{ij}N$

$$\Gamma''(t)=\underbrace{\left((u^k)''+C_{ij}^k(u^i)'\cdot(u^j)'\right)P_k}_{\text{Total tangential component}}+\underbrace{(u^i)'\cdot(u^j)'L_{ij}N}_{\text{Normal component}}$$

Christoffel symbols

- Used to express curvatures (or second derivatives) in the 3D euclidean space from curvatures (or second derivatives) expressed in the parametric space.
- Normal curvature: based on the normal component
  $$(u^i)' \cdot (u^j)' L_{ij} N$$
  If we have an arc-length parametrization :
  $$\kappa_n = \Gamma'' \cdot N = L_{ij} (u^i)' (u^j)' \qquad \left(= \varphi_2 (d\Gamma^{uv}, d\Gamma^{uv}) \right)$$
  if not : need to adjust
  $$\kappa_n = \frac{L_{ij} (u^i)' (u^j)'}{ds^2} = \frac{L_{ij} (u^i)' (u^j)'}{g_{kl} (u^k)' (u^l)'}$$
- One recovers the formula : $\kappa_N = \dfrac{\varphi_2 (d\Gamma^{uv}, d\Gamma^{uv})}{\varphi_1 (d\Gamma^{uv}, d\Gamma^{uv})}$

Christoffel symbols

- Geodesic curvature : based on the tangential components

$$\left(\left(u^k\right)'' + C^k_{ij}\left(u^i\right)'\cdot\left(u^j\right)'\right)P_k$$

- If the curve is parametrized by arc-length :

$$\Gamma'\cdot\Gamma' = 1 \qquad \Gamma'\cdot\Gamma'' = 0$$

$$\Gamma''\cdot\Gamma' = 0 = \boxed{\left(\left(u^k\right)'' + C^k_{ij}\left(u^i\right)'\cdot\left(u^j\right)'\right)P_k}\cdot\Gamma' + \left(u^i\right)'\cdot\left(u^j\right)' L_{ij}\,\overset{=0}{N\cdot\Gamma'}$$

- Therefore, the tangential component Γ_t'' of Γ'' is orthogonal to Γ' and to N as well. There is a proportionality factor between Γ_t'' and $N\times\Gamma'$. This factor is the geodesic curvature

Christoffel symbols

- $\Gamma_t{}'' = \kappa_g (N \times \Gamma')$ $\quad \Gamma_t{}'' \cdot (N \times \Gamma') = \kappa_g (N \times \Gamma') \cdot (N \times \Gamma')$

  $\kappa_g = (N \times \Gamma') \cdot \Gamma_t{}'' = (N \times \Gamma') \cdot \Gamma''$

  Therefore, $\kappa_g = [N, \Gamma', \Gamma'']$, valid only if the parametrization of Γ is normal (arc-length)

- If not, need to adjust ...

  $$\kappa_g = \frac{[N, \Gamma', \Gamma'']}{ds^3} = \frac{[N, \Gamma', \Gamma'']}{\left(g_{kl} (u^k)' (u^l)'\right)^{3/2}}$$

- Some developments* leads to the following identity :

$$\kappa_g = \frac{\sqrt{det(M_1)} [-C_{11}^2 u'^3 + C_{22}^1 v'^3 - (2C_{12}^2 - C_{11}^1) u'^2 v' + (2C_{12}^1 - C_{22}^2) u' v'^2 + u'' v' - v'' u']}{\left(\begin{pmatrix} u' & v' \end{pmatrix} \mathrm{M}_1 \begin{pmatrix} u' \\ v' \end{pmatrix} \right)^{3/2}}$$

Again, this expression makes only use of derivatives in the parametric space...

* see e.g."Modern Differential Geometry of Curves and Surfaces with Mathematica", 2nd ed. Boca Raton, FL: CRC Press, pp. 501-518, 1997.

Christoffel symbols

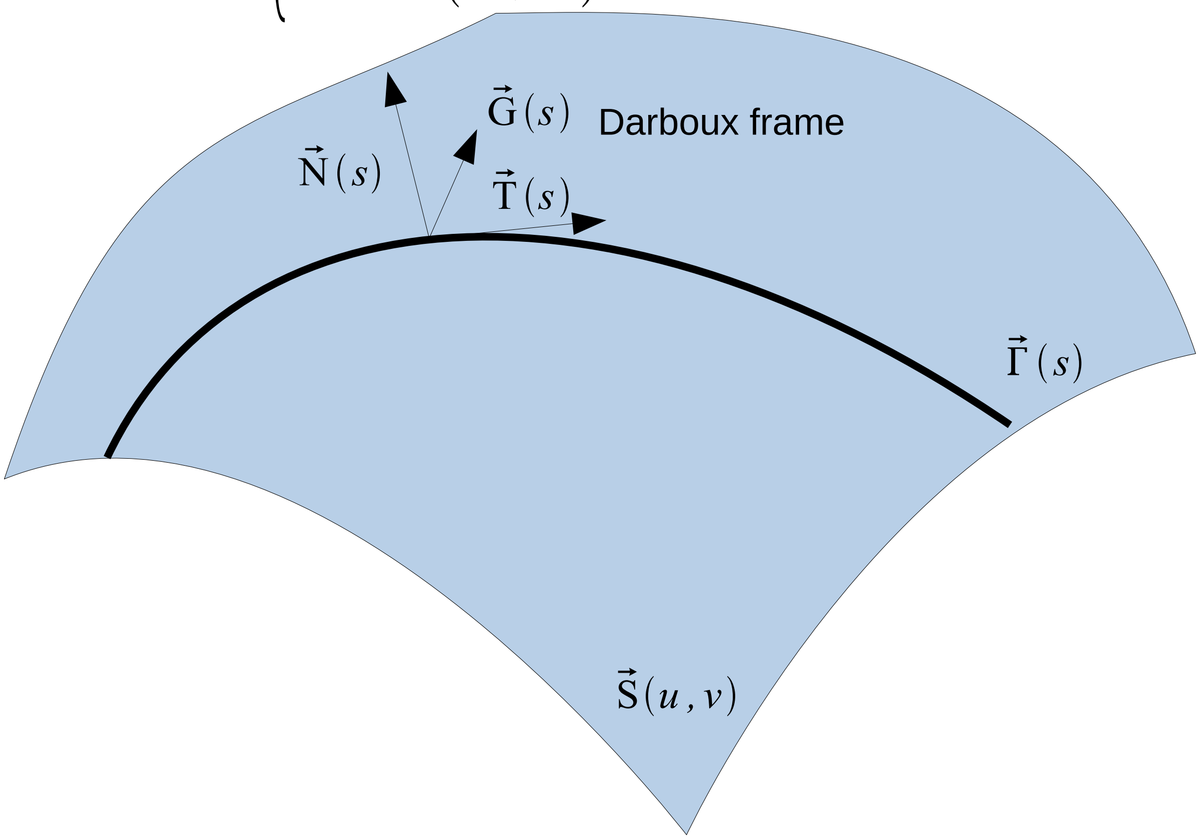
- Equations of a geodesic

    Need to set that the geodesic curvature is equal to zero.

    - One obtains a set of differential equations
    - May be used to compute an approximation of the geodesic since it is usually impossible to solve that coupled system of ODEs algebraically.

Darboux frame and equations

$$\vec{P}(u,v):\begin{cases} x=f(u,v) \\ y=g(u,v) \\ z=h(u,v) \end{cases} \quad \vec{\Gamma}^{uv}(s):\begin{cases} u=u(s) \\ v=v(s) \end{cases} \quad \vec{\Gamma}(s):\begin{cases} x(s)=f(u(s),v(s)) \\ y(s)=g(u(s),v(s)) \\ z(s)=h(u(s),v(s)) \end{cases}$$



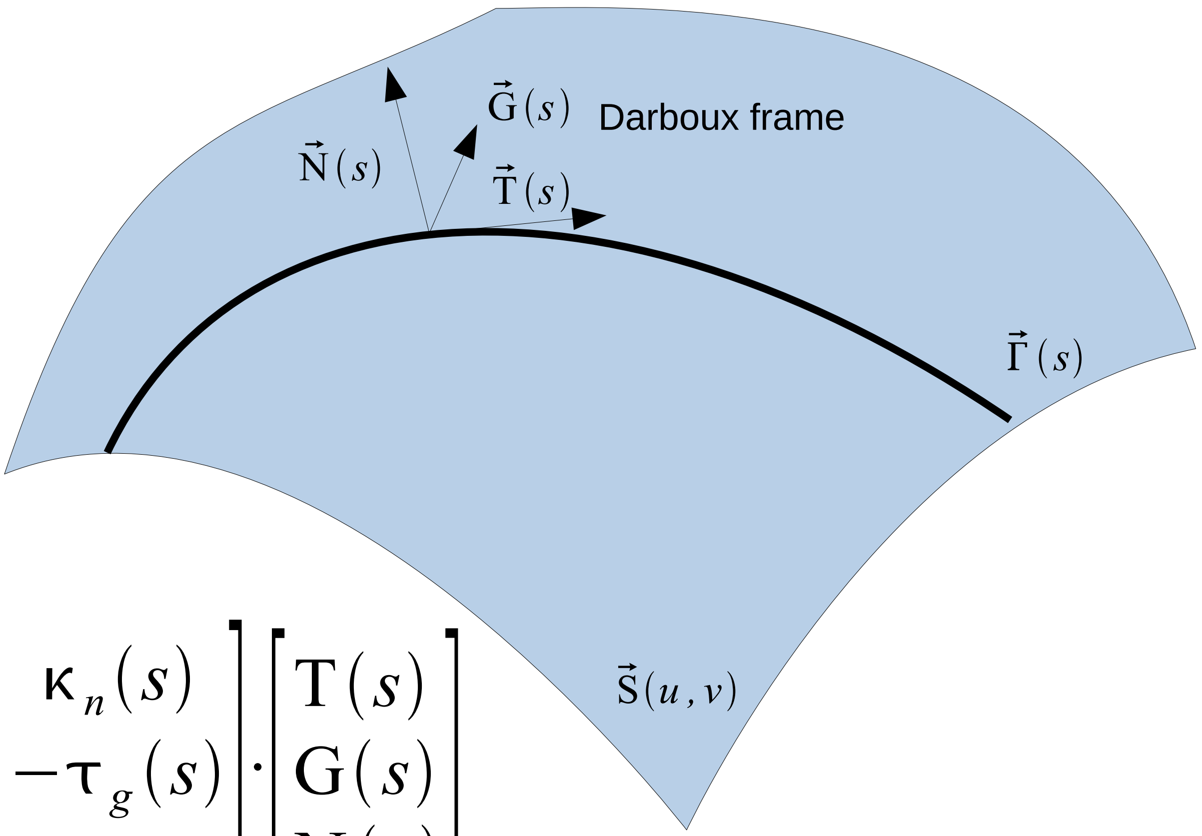
$$\vec{T}(s)=\left\{\frac{\partial x(s)}{\partial s},\frac{\partial y(s)}{\partial s},\frac{\partial z(s)}{\partial s}\right\}$$

$$\vec{N}(s)=\vec{N}(u(s),v(s))$$

$$\vec{G}(s)=\vec{N}\times\vec{T}$$

s is the curvilinear abscissa

Darboux frame and equations



$$\begin{bmatrix} \dfrac{d\,T(s)}{ds} \\ \dfrac{d\,G(s)}{ds} \\ \dfrac{d\,N(s)}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_g(s) & 0 & -\tau_g(s) \\ -\kappa_n(s) & \tau_g(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ G(s) \\ N(s) \end{bmatrix}$$

Geodesic torsion (or relative torsion)

Parametric representation

- Geodesic torsion

  is identically zero iff the curve is along a direction of principal curvature (line of curvature)

- Geodesic curvature

  is identically zero iff the curve is a geodesic

Parametric representation

- In an euclidean space:
  - The shortest line between two points has simple curvature κ equal to zero (= straight line)
  - The minimal surface carried by a curve is of average curvature κ_M equal to 0 everywhere
- In a non euclidean space (on a surface):
  - The (or one of the) shortest line between two points on the surface has a geodesic curvature κ_g equal to zero (it's a geodesic)

Parametric representation

- A good reference on « shape interrogation »
  - Shape interrogation is the ability to get information from geometric models

N. M. Patrikalakis and T. Maekawa, *Shape Interrogation for Computer Aided Design and Manufacturing,* Springer Verlag, February 2002.
ISBN 3-540-42454-7

(Available as a PDF electronic copy at the library)

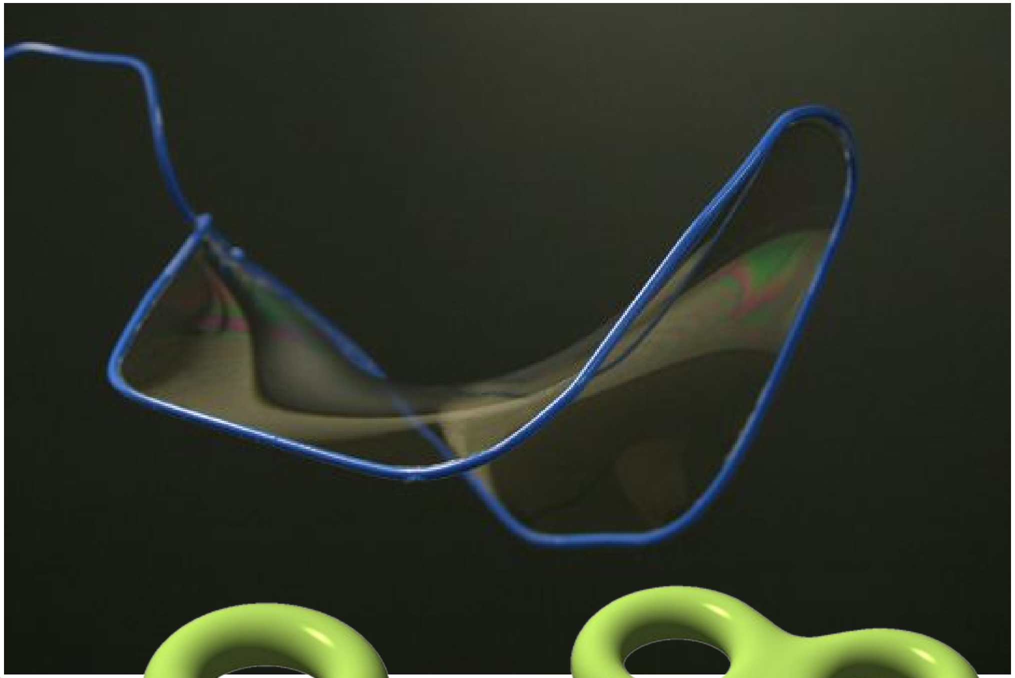
Parametric representation

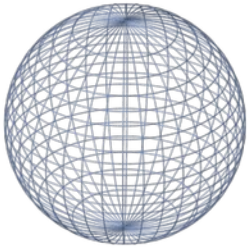
Some results of differential geometry

Gauss-Bonnet's theorem

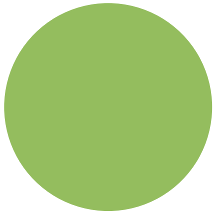
$$\int_S \kappa_T \, dA + \int_{\partial S} \kappa_g \, ds = 2\pi\chi(S)$$

$\chi(S)$ is Euler's characteristic of surface S





$\chi=2$



$\chi=1$

$\chi=0$

$\chi=-2$

$\chi=-4$

Application of Gauss-Bonnet's theorem

- For a polyhedron, we have (Euler's formula) :

$$\chi(S)=N-E+F$$

We want to approximate a sphere with hexagons (H) and pentagons (P).

- Each pentagon (hexagon) has 5 (6) nodes, shared by 3 faces
- Each pentagon (hexagon) has 5 (6) edges, shared by 2 faces
- So the total number of nodes N is $(5P+6H)/3$
- Total number of edges E is $(5P+6H)/2$, total number of faces F is $P+H$

$$\chi(S)=N-E+F=(5\mathrm{P}+6\mathrm{H})/3-(5\mathrm{P}+6\mathrm{H})/2+P+H=P/6$$

- The sphere is compact and its gaussian curvature is $1/R^2$

$$\int_S \kappa_T \, dA+\int_{\partial S} \kappa_G \, ds=\frac{1}{R^2}\int_S dA=4\pi=2\pi\chi(S)$$

so $P=12$ … 12 pentagons are necessary and as many hexagons as we want to make soccer ball.

Parametric representation